Homework 1: The Nature of Proof

Mathematics Specialist 1&2

Name: ______________________________

Date: _____________

Instructions: Answer all questions. Show all working clearly. For proof questions, write your reasoning in full sentences.

Part A Direct Proof [20 marks]

Q1. Let m and n be integers. Assume that m is odd and n is odd. Prove that $m + n$ is even. [2]

Q2. Suppose that m is divisible by 3 and n is divisible by 7. Prove that mn is divisible by 21. [2]

Q3. Suppose that m and n are both perfect squares. Show that mn is also a perfect square. [3]

Q4. Let m and n be integers. Prove that $(m + n)^2 - (m - n)^2$ is divisible by 4. [3]

Q5. Suppose that n is an odd integer. Prove that $n^2 + 8n + 3$ is even. [3]

Q6. Let $x, y \in \mathbb{R}$. Show that $x^2 + y^2 \geq 2xy$. [3]

Q7. Suppose that m is divisible by 3 and n is divisible by 7. Prove that m^2n is divisible by 63. [4]

Part B Negation and Contrapositive [24 marks]

Q8. Write down each statement and its negation. State which is true and which is false. [4]

(a) $1 > 0$

(b) 4 is divisible by 8

(c) Each pair of primes has an even sum.

(d) Some rectangle has four sides of equal length.

Q9. Write down each statement and its negation (using De Morgan's Laws where needed). State which is true and which is false. [4]

(a) 14 is divisible by 7 and 2.

(b) 12 is divisible by 3 or 4.

(c) 15 is divisible by 3 and 6.

(d) 10 is divisible by 2 or 3.

Q10. Write down the contrapositive of each statement. [4]

(a) If it is raining, then there are clouds in the sky.

(b) If you are smiling, then you are happy.

(c) Let $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

(d) Let $m, n \in \mathbb{Z}$. If $m+n$ is even, then m and n are either both even or both odd.

Q11. For each statement below, write down and **prove** the contrapositive. [12]

(a) Let $n \in \mathbb{Z}$. If $3n + 5$ is even, then n is odd.

(b) Let $n \in \mathbb{Z}$. If $n^2 - 8n + 3$ is even, then n is odd.

(c) Let $n \in \mathbb{Z}$. If $n^3 + 1$ is even, then n is odd.

Part C Equivalent Statements [20 marks]

Q12. Write down and prove the **converse** of each statement. [8]

(a) Let $n \in \mathbb{Z}$. If n is odd, then $n - 3$ is even.

(b) Let $n \in \mathbb{Z}$. If n^2 is divisible by 5, then n is divisible by 5.

Q13. Determine whether each pair of statements P and Q is **equivalent**. Justify your answer. [4]

(a) P: $2x = 4$ Q: $x = 2$

(b) P: m is even or n is even, where $m, n \in \mathbb{Z}$ Q: mn is even, where $m, n \in \mathbb{Z}$

Q14. Let n be an integer. Prove that $n + 1$ is odd **if and only if** $n + 2$ is even. [4]

Q15. Let n be an integer. Prove that n^3 is even **if and only if** n is even. [4]

Part D Counterexamples and Disproving [16 marks]

Q16. Classify each statement as a **universal** statement ('for all') or an **existence** statement ('there exists'). [4]

(a) Every natural number greater than 1 has a prime factorisation.

(b) Some natural numbers are prime.

(c) At least one real number x satisfies $x^2 - x - 1 = 0$.

(d) Any positive real number has a square root.

Q17. Write down the **negation** of each statement. [4]

(a) For every natural number n, the number $2n^2 - 4n + 31$ is prime.

(b) There exist $x, y \in \mathbb{R}$ such that $x < y$ and $x^2 > y^2$.

(c) For all $x, y \in \mathbb{R}$, we have $(x + y)^2 = x^2 + y^2$.

Q18. Find a **counterexample** to disprove each statement. [4]

(a) For all $x \in \mathbb{R}$, we have $x^2 > x$.

(b) Let $n \in \mathbb{Z}$. If $n^3 - n$ is even, then n is even.

(c) If $m, n \in \mathbb{N}$, then $m + n \leq mn$.

Q19. Show that each existence statement is **false** by proving its negation. [4]

(a) There exists $n \in \mathbb{N}$ such that $9n^2 - 1$ is a prime number.
Hint: Factorise $9n^2 - 1$.

(b) There exists $x \in \mathbb{R}$ such that $2 + x^2 = 1 - x^2$.

End of Homework 1. Total: 80 marks.