



Essential Vectors Exercise Set

Specialist Mathematics – Vectors I to VI & Geometric Proofs

WACE Topic: Specialist Year 12 – Vectors (Algebra, Dot/Cross Product, Lines, Planes, Spheres, Vector Functions, Proofs)

Vectors I – Vector Algebra, Dot and Cross Product

Calculator Free

Question 1

[8 marks: 2, 1, 1, 4]

Given that $\mathbf{a} = 3\mathbf{i} + 10\mathbf{j} + 8\mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} + 3\mathbf{k}$, find:

- (a) $|\mathbf{a} - 2\mathbf{b}|$
- (b) the unit vector parallel to $\mathbf{a} - 2\mathbf{b}$
- (c) a vector that is parallel to $\mathbf{a} - 2\mathbf{b}$ but with a magnitude of 10
- (d) $\mathbf{a}$ in terms of $\mathbf{p}$ and $\mathbf{q}$ where $\mathbf{p} = -\mathbf{i} + 4\mathbf{k}$ and $\mathbf{q} = 2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$.

Question 2

[6 marks]

$\mathbf{OA} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{OB} = 2\mathbf{i} + b\mathbf{j} - \mathbf{k}$ and $\mathbf{OC} = 6\mathbf{i} + 3\mathbf{j} + c\mathbf{k}$. Find b and c if A, B and C are collinear.

Question 3

[5 marks: 3, 1, 1]

- (a) Find the acute angle between the vectors $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + \mathbf{j}$.

Hence, or otherwise, find the acute angle between:

- (b) $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $-\mathbf{i} - \mathbf{j}$

- (c) $-4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ and $-5\mathbf{i} - 5\mathbf{j}$.

Question 4

[6 marks: 3, 3]

- (a) Find a unit vector that is parallel to $-\mathbf{i} + \mathbf{j} - \mathbf{k}$ and perpendicular to $-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
- (b) Find a unit vector that is perpendicular to both $-\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Question 5

[8 marks: 3, 1, 4]

Let $\mathbf{u} = \langle 2, 1, 2 \rangle$, $\mathbf{v} = \langle 1, 0, 1 \rangle$ and $\mathbf{w} = \langle -1, 1, 0 \rangle$.

- (a) Use the result $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$, where θ is the acute angle between $\mathbf{a}$ and $\mathbf{b}$, to calculate the sine of the angle between $\mathbf{v}$ and $\mathbf{w}$.
- (b) Determine the area of the parallelogram with sides parallel to $\mathbf{v}$ and $\mathbf{w}$.
- (c) The scalar projection of vector $\mathbf{a}$ onto vector $\mathbf{b}$ is given by $\mathbf{a} \cdot \hat{\mathbf{b}}$. Find the vector projection of $\mathbf{u}$ onto $\mathbf{v} \times \mathbf{w}$.

Vectors II – Position Vectors and Geometry in 3D

Calculator Free**Question 6**

[9 marks: 3, 3, 3]

The points A and B have position vectors $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $-3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ respectively.

- (a) Find the position vector of the point K which is 6 units from A in the direction $-2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.
- (b) Find the position vector of the point L which is 21 units from A in the direction $\mathbf{AB}$.
- (c) Find the area of $\triangle KAL$. Show clearly how you obtained your answer.

Question 7

[10 marks: 3, 3, 4]

Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{k}$ and $\mathbf{w} = \langle 2, -2, 2 \rangle$.

- (a) Find the acute angle between $\mathbf{u}$ and $\mathbf{v}$.
- (b) Find the area of a triangle with sides parallel and congruent to $\mathbf{u}$ and $\mathbf{v}$.
- (c) A triangular pyramid has sides parallel and congruent to $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Determine the volume of this triangular pyramid.

Calculator Assumed**Question 8**

[11 marks: 2, 2, 4, 3]

A rectangular box OABCDEFGH rests on the x - y plane. The vertex O has position vector $\langle 0, 0, 0 \rangle$ cm. $CB = 10$ cm, $AB = BE = 8$ cm.

- (a) State the position vectors of the vertices C and E of this box.
- (b) Use vector methods to find the length of the diagonal BG.
- (c) Use a cross-product method to calculate exactly $\sin(\angle GBF)$.
- (d) Calculate the exact area of $\triangle BGF$.

Vectors III – Lines**Calculator Free****Question 9**

[7 marks: 2, 2, 3]

The parametric equation of a line is given by $x = \frac{1 + \lambda}{2}$, $y = \frac{-1 + 2\lambda}{2}$, $z = \frac{4 - 3\lambda}{5}$.

- (a) Determine the Cartesian equation of this line.

- (b) Find the vector equation of this line.
- (c) The points A, B and C on this line are such that $\lambda = -1$, $\lambda = 0$ and $\lambda = 1$ respectively. Find the ratio with which the point B divides the line AC.

Calculator Assumed

Question 10

[9 marks: 2, 4, 3]

The lines L_1 and L_2 respectively have equations $\mathbf{r} = \langle 1, 1, 2 \rangle + \lambda \langle -2, 1, -3 \rangle$ and $\mathbf{r} = \langle 2, 1, 4 \rangle + \mu \langle 1, -1, 1 \rangle$. The lines L_1 and L_2 intersect at K.

- (a) Find the acute angle between L_1 and L_2 .
- (b) Use a vector method to find the position vector of K.
- (c) Find the vector equation of the line passing through the point K and perpendicular to L_1 and L_2 .

Question 11

[10 marks: 4, 2, 2, 2]

The line L_1 has equation $\mathbf{r} = \langle -4 + \lambda, -2\lambda, \lambda - 2 \rangle$. The point B $(-4, 0, -2)$ lies on the line L_1 . The point A has position vector $\langle 2, -2, 0 \rangle$.

- (a) Find the position vector of the point K on the line L_1 such that AK is perpendicular to L_1 .
- (b) Hence, find the shortest distance between A and the line L_1 .
- (c) Find the size of $\angle BAK$.
- (d) Find the area of $\triangle AKB$.

Question 12

[9 marks: 4, 4, 1]

The point A $(1, 0, -2)$ lies on the line L_1 which is parallel to $\mathbf{d}_1 = \langle 4, -2, -4 \rangle$. The point B $(6, -1, -10)$ lies on the line L_2 which is parallel to $\mathbf{d}_2 = \langle 2, 1, 2 \rangle$.

- (a) Prove that the lines L_1 and L_2 are neither parallel nor intersecting.
- (b) Determine the scalar projection of $\mathbf{BA}$ onto $\mathbf{d}_1 \times \mathbf{d}_2$.
- (c) Hence, find the shortest distance between the lines L_1 and L_2 .

Vectors IV – Planes

Calculator Free

Question 13

[7 marks: 3, 4]

The points with position vectors $\langle 1, 0, 2 \rangle$, $\langle -1, 1, 0 \rangle$ and $\langle 0, 1, -1 \rangle$ lie on a plane.

- (a) Determine the vector equation of the plane in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$.
- (b) Determine the Cartesian equation of the plane.

Calculator Assumed

Question 14

[9 marks: 1, 1, 2, 2, 3]

The points A and B have position vectors $\langle 1, -2, 1 \rangle$ and $\langle 0, 0, 1 \rangle$ respectively. The plane Π has equation $\mathbf{r} \cdot \langle 2, -1, 1 \rangle = 1$.

- (a) Find a unit vector perpendicular to the plane Π .
- (b) Show that the point B lies on the plane Π .
- (c) Find $|\mathbf{BA}|$.
- (d) Find the acute angle between $\mathbf{BA}$ and the normal to the plane Π .

- (e) Hence, find the minimum distance between the point A and the plane Π .

Question 15

[11 marks: 2, 3, 3, 3]

The plane P has equation $\mathbf{r} \cdot (2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) = -2$. The line L has equation $\frac{x}{-1} = \frac{y-2}{1} = \frac{z+1}{2}$.

- (a) Find the vector equation for line L.
- (b) Show that the line L does not intersect the plane P.
- (c) Find the Cartesian equation of the plane Q containing line L and parallel to P.
- (d) Determine with reasons if the planes P and Q are on the same side or on opposite sides of the origin $(0, 0, 0)$.

Vectors V – Spheres

Calculator Free**Question 16**

[10 marks: 2, 2, 2, 4]

The sphere S has centre $(1, -1, 2)$ and radius 5. The point P with coordinates $(5, 2, 2)$ lies on this sphere.

- (a) Determine the vector and Cartesian equations of this sphere.
- (b) PQ is a diameter of this sphere, determine the coordinates of Q.
- (c) The sphere intersects the y - z plane in the form of a circle. Determine the Cartesian equation of this circle.
- (d) Determine the value(s) of k if the point with position vector $\langle 1, 3, k \rangle$ is outside this sphere.

Calculator Assumed**Question 17**

[10 marks: 4, 6]

The line with equation $x - 1 = \frac{y - 2}{2} = 1 - z$ intersects the sphere with equation $x^2 + y^2 + z^2 + 2x + 4y - 2z - 30 = 0$ at the points A and B.

- (a) Determine the vector equations of the line and the sphere.
- (b) Hence, or otherwise, determine the exact length of the chord AB.

Question 18

[8 marks: 4, 4]

Consider the sphere S with equation $|\mathbf{r} - \langle 1, 1, 1 \rangle| = 3$.

- (a) Prove that the line L with equation $\mathbf{r} = \langle 3, -1, 2 \rangle + \lambda \langle 1, 1, 0 \rangle$ is a tangent to the sphere S.
- (b) Prove that the plane with equation $\mathbf{r} \cdot \langle 2, 2, 1 \rangle = 14$ is tangential to S.

Vectors VI – Vector Functions**Calculator Assumed****Question 19**

[5 marks]

The position vectors of particles A and B at time t seconds are respectively

$$\mathbf{r}_A = \begin{pmatrix} 2 + t \\ 3 - t \\ -2 + 4t \end{pmatrix} \quad \text{and} \quad \mathbf{r}_B = \begin{pmatrix} -4 + 3t \\ 4 - 2t \\ -1 + 7t \end{pmatrix}.$$

Find when and where A crosses the path of B.

Question 20

[11 marks: 2, 3, 3, 3]

An eagle flies with a constant velocity of $\langle 2, 4, 1 \rangle$ metres per minute. At 7 am, the eagle flies past the point A with position vector $\langle 10, -10, 2 \rangle$. Find:

- (a) the position vector of the eagle at 7.15 am (same day)

- (b) when the eagle is 50 m from A
- (c) when and where the eagle crosses the x - z plane
- (d) using a vector method, the closest distance between the eagle and the point B with position vector $\langle 20, 5, 8 \rangle$ m.

Question 21*[13 marks: 3, 4, 2, 4]*

The position vectors of objects P and Q at time t seconds are given by

$$\mathbf{OP}(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ metres} \quad \text{and} \quad \mathbf{OQ}(t) = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \text{ metres respectively.}$$

- (a) Show that P and Q do not collide.
- (b) Calculate the closest distance between P and Q.
- (c) State the vector equations of the paths of P and Q.
- (d) Calculate the closest distance between the paths of P and Q.

Geometric Proofs Using Vectors

Calculator Assumed**Question 22***[8 marks: 1, 3, 4]*

OAB is a right angled triangle with $\angle AOB = 90^\circ$. $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. M is the midpoint of AB.

- (a) Explain why $\mathbf{a} \cdot \mathbf{b} = 0$.
- (b) Find $|\mathbf{BM}|^2$ in terms of a and b , where $|\mathbf{a}| = a$ and $|\mathbf{b}| = b$.

- (c) Hence, prove that M is the centre of a circle passing through A, B and O.

Question 23*[7 marks: 2, 5]*

The accompanying diagram shows a tetrahedron. Two of the adjacent edges of the tetrahedron are represented by the vectors $\mathbf{a}$ and $\mathbf{b}$. A third edge is represented by the vector $\mathbf{d}$.

- (a) Prove that the area of the face bounded by $\mathbf{a}$ and $\mathbf{b}$ is given by $A = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

- (b) Prove that the volume of the tetrahedron is given by $V = \frac{1}{6}|\mathbf{d} \cdot (\mathbf{a} \times \mathbf{b})|$.

Question 24*[13 marks: 2, 2, 5, 3, 1]*

OAB is a triangle with $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. D, E and F are the midpoints of OB, AB, and OA respectively. $\mathbf{AM} = \alpha\mathbf{AD}$ and $\mathbf{MF} = \beta\mathbf{BF}$.

- (a) Find $\mathbf{AD}$ and $\mathbf{BF}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
- (b) Find $\mathbf{AM}$ and $\mathbf{MF}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
- (c) Use your answers in (b) to find α and β .
- (d) Show that $\mathbf{OM} = \mu\mathbf{OE}$, giving the value of μ .
- (e) Comment on the significance of the location of M in terms of the lines OE, AD and BF.