30 Conjectures & Proofs

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1. [10 marks: 2, 2, 3, 3]

Provide a counter-example to disprove each of the following conjectures.

(a) If $x^2 > 100$, then $x > 10$.

(b) If $2 > x$, then $2^2 > x^2$.

(c) The sum of two numbers is always greater than the larger of the two numbers.

(d) The sum of any two positive numbers must always be less than the product of these two numbers.

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2. [5 marks: 1, 1, 3]

(a) Given that $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$. Find:

(i) $\mathbf{P} \times \mathbf{Q}$

(ii) $\mathbf{Q} \times \mathbf{P}$

(b) Prove or disprove the conjecture that if the matrices **P** and **Q** are such that $\mathbf{P} \times \mathbf{Q} = \mathbf{I}$, where **I** is the relevant identity matrix, then $\mathbf{P}^{-1} = \mathbf{Q}$.

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3. [9 marks: 3, 3, 3]

Provide a counter-example to disprove each of the following conjectures.

(a) Consider the parallelograms ABCD and PQRS. If the diagonals AC = PR then ABCD and PQRS are congruent.

(b) Consider the parallelograms ABCD and PQRS. If AB = PQ and the diagonals AC = PR, ABCD and PQRS are congruent.

(c) If the diagonals of a quadrilateral are perpendicular then the quadrilateral must be a rhombus.

Calculator Assumed

4. [12 marks: 2, 2, 4, 4]

Provide a counter-example to disprove each of the following conjectures.
Show all attempts, successful and otherwise.

(a) $2^{2n} + 1$ is prime for integer $n \geq 1$.

(b) $n^2 - n + 5$ is prime for integer $n \geq 1$.

(c) $1^n + 5^n + 10^n + 18^n + 23^n + 27^n = 2^n + 3^n + 13^n + 15^n + 25^n + 26^n$ for integer $n \geq 1$.

(d) There are no integer solutions to $a^3 + b^3 = c^3 + d^3$ where $a \neq b \neq c \neq d$.

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5. [3 marks]

Prove that $2^n - 1$ is always odd for integer $n \geq 1$.

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6. [3 marks]

Prove that the square of an odd number add 11 is a multiple of 4.

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7. [5 marks: 2, 3]

(a) Prove that product of three consecutive integers is divisible by 3.

(b) Prove that product of three consecutive integers is divisible by 6.

Calculator Assumed

8. [3 marks]

Prove that $x^7 - x$ is divisible by 6 for integer $x \geq 1$.

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9. [4 marks]

Prove that the product of any three consecutive even numbers must be a multiple of 24.

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10. [3 marks]

Prove that $4n^3 - 4n$ is a multiple of 24 for integer $n \geq 1$.

Calculator Assumed

11. [7 marks: 4, 3]

(a) It is conjectured that a number is divisible by 4 if the sum of twice the tens digit and the ones digit is a multiple of 4. Prove that this conjecture is true for a four digit number.

(b) Consider the positive integers a and b. The arithmetic mean of these two integers is $M = \dfrac{a+b}{2}$ and the harmonic mean is $H = \dfrac{2ab}{a+b}$.
It is conjectured that $M \geq H$. Use the expansion of $(\sqrt{a} - \sqrt{b})^2$ to prove this conjecture.

Calculator Assumed

12. [7 marks: 2, 5]

(a) Provide a counter-example to disprove the conjecture that the cube of an even number greater than two less the number itself is divisible by 5.

(b) Prove that $x^5 + x^4 + x^3 + x^2$ is divisible by 4 for x as a whole number.

Calculator Assumed

13. [7 marks: 1, 1, 5] [TISC]

Consider the conjecture that the product of any two positive numbers must be greater than the sum of these two numbers.

(a) Provide an example that proves this conjecture.

(b) Provide an example that disproves this conjecture.

(c) Under what conditions will this conjecture be always true?

Calculator Assumed

14. [7 marks: 2, 2, 3] [TISC]

Consider the conjecture that the *quotient* of any two positive numbers must be less than the *product* of these two numbers. That is: if one positive number is divided by another positive number, then the result must be less than the *product* of these two numbers.

(a) Provide an example that supports this conjecture.

(b) Provide an example that disproves this conjecture.

(c) Under what conditions will this conjecture be always true?
Justify your answer.

Calculator Assumed

15. [10 marks: 3, 3, 4]

(a) Prove that $15.\overline{15}$ is a rational number.

(b) Prove that $5.7\overline{35}$ is a rational number.

(c) Prove that $10 + \dfrac{1}{10} + \dfrac{1}{10^2} + \dfrac{1}{10^3} + \dfrac{1}{10^4} + \ldots$ is a rational number.

Calculator Assumed

16. [9 marks: 5, 4]

(a) State the negative of the converse of the following conjecture:

If n is a multiple of 10, then n^2 is a multiple of 10.

Prove that the converse is true by proving that the negative of the converse is false.

(b) Use the result from (a) and the method of contradiction to prove that $\sqrt{10}$ is an irrational number.

Calculator Assumed

17. [10 marks: 5, 5]

(a) Consider the conjecture:

If m is a rational number and n is an irrational number,
then $m + n$ is an irrational number.

State the negative of this conjecture.
Prove that this conjecture is true by proving that its negative is false.

(b) Prove that if m is a rational number and n is an irrational number,
then $m \times n$ is an irrational number.

Calculator Assumed

18. [7 marks: 5, 2]

(a) Use mathematical induction to prove that:

$$\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots. + \frac{n}{5^n} = \frac{5}{16} - \frac{4n+5}{16(5^n)} \text{ for integer } n \geq 1.$$

(b) Discuss the sum as the number of terms increases indefinitely.

Calculator Assumed

19. [5 marks]

Prove inductively that $(1 + i)^{4n} = (-1)^n\, 2^{2n}$ for integer $n \geq 1$.

20. [5 marks]

Use mathematical induction to prove that $11^n + 4$ is divisible by 5 for integer $n \geq 1$.

Calculator Assumed

21. [6 marks]

Prove that $10^{n+1} + 3 \times 10^{n} + 5$ is a multiple of 9 for positive integer n.

22. [4 marks]

Given the non-singular commutative matrices **P** and **Q**, use mathematical induction to prove that for integer $n \geq 1$, $\mathbf{P}^{n} = \mathbf{Q}\,\mathbf{P}^{n}\,\mathbf{Q}^{-1}$.

Calculator Assumed

23. [5 marks] [TISC]

Use mathematical induction to prove that for integer $n \geq 1$:

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos (2n - 1)x = \frac{\sin 2nx}{2 \sin x}.$$

[Hint: Use the formula $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$.]

Calculator Assumed

24. [4 marks]

Complete the table below.

Conjecture	
Negative of conjecture	
Contrapositive of conjecture	
Converse of conjecture	If x is an odd number then x^2 is an odd number.
Inverse of conjecture	

25. [5 marks: 2, 3]

Consider the conjecture:

If a is a factor of 20, then a is also factor of 40.

(a) Prove that the conjecture is true.

(b) State the inverse of the conjecture and determine with reasons whether it is true or false.

Calculator Assumed

26. [7 marks: 5, 2]

Consider the statement: $\tan\theta = 0 \quad \Rightarrow \quad \sin\theta = 0.$

(a) Determine with reasons if the contrapositive of this statement is true or false.

(b) Hence or otherwise, determine with reasons if the conjecture is true of false.

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27. [6 marks: 3, 3]

Consider the statement: $\cos\theta = \frac{1}{2} \quad \Rightarrow \quad \tan\theta = \sqrt{3}$

(a) Determine with reasons if the converse of this statement is true or false.

(b) Hence, or otherwise determine with reasons if the inverse is true or false.